MODELS OF TURBULENCE

- Mixing Length Models
- One Equation Models
- Two-Equation Models

} Isotropic Turbulence

- Reynolds Stress Model
- Large Eddy Simulation (LES)

} Anisotropic Effects

- Direct Numerical Simulation (DNS)

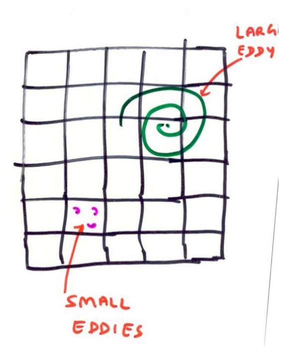
In a turbulent flow, eddies of various sizes exist. For example, in a typical lab scale experiment, eddies with a size range of microns to centimetres may be observed. These eddies cause fluctuations, which have time constants of the order of μs to seconds.

1. DIRECT NUMERICAL SIMULATION (DNS)

In this approach, the idea is to capture the flow effects of all the eddies faithfully, by employing grid sizes < μs. For any practical problem, the number of grid points required may be of the order of 10^{12} and several million time steps may be required. Meeting such a heavy computational demand is not feasible in many institutions. Extremely fast and powerful parallel computers are needed to perform DNS.

2. LARGE EDDY SIMULATION (LES)

This approach divides the eddies into two categories large which need to be precisely modelled and small which can be approximately accounted for. A moderately dense grid (few million points) is selected which captures large eddies exactly. For eddies with size less than grid size, approximate sub-grid scale models are evoked.



3. REYNOLDS STRESS MODEL (RMS)

Transport equations are derived for the Reynolds stress and for turbulent fluxes.

Reynolds stresses:

$$-\rho\,\overline{u'^2},\quad -\rho\,\overline{v'^2},\quad -\rho\,\overline{w'^2}$$

$$-\rho\,\overline{u'v'},\quad -\rho\overline{u'w'},\quad -\rho\,\overline{v'w'}$$

Turbulent Fluxes:

(For Heat Transfer)

$$-\rho C_p\,\overline{u'T'},\quad -\rho C_p\overline{v'T'},\quad -\rho C_p\,\overline{w'T'}$$

For each of the above quantities, a transport equation can be obtained. The RMS approach involves 6 Additional differential equations to be solved for the 6 Reynolds stresses and more equations for the turbulent fluxes. Although it involves more computations, this model can capture the anisotropic effects in the turbulent flow. Grid pints used are not as large as in the cases of DNS and LES. Hence it is computationally more economical than DNS and LES.

LOWER ORDER MODELS

In these models, the fluctuations due to eddies are not precisely simulated. The Stresses and Fluxes are represented as:

$$\tau_{xy} = -\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$$

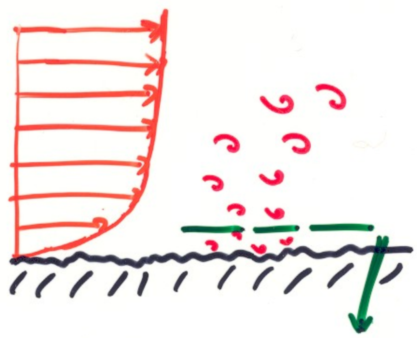
$$q_y = -k\frac{\partial T}{\partial x} \quad \text{etc.}$$

where

$\mu = \mu_l + \mu_t$ $\mu_l \rightarrow$ Laminar Viscosity $\mu_t \rightarrow$ Turbulent or Eddy Viscosity

$k = k_l + k_t$ $k_l \rightarrow$ Laminar Conductivity $k_t \rightarrow$ Turbulent Conductivity

μ_t, k_t arise because of mixing enhancement by eddies. While μ_l, k_l depend on the type of fluid (and the local temperature, concentration etc.) μ_t, k_t depend on eddy properties. They vary significantly from location to location



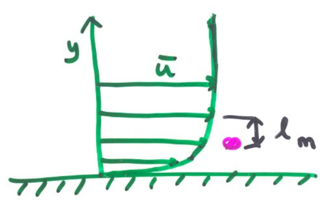
In Laminar Sublayer, $\mu_t << \mu_l$.

Far away from wall, $\mu_t >> \mu_l$.

In between, $\mu \approx \mu_t$.

<u>MIXING LENGTH MODEL (PRANDTL)</u>

Turbulent Diffusivity, $\nu_t = \mu_t/s$. In analolgy with kinetic theory of molecules, which gives $\nu_l \cong \lambda \cdot v_{rms}$, where $\lambda \rightarrow$ mean free path, and $v_{rms} \rightarrow$ root mean square velocity of molecules, Prandtl Proposed $\nu_t \sim l_m \cdot v'$, where and v' the (Root Mean Square) fluctuation velocity of an eddy and mixing length of an eddy l_m is the average distance travelled by an eddy before it gives up its momentum completely by mixing with main flow.



Prandtl Proposed $v' = l_m \cdot \frac{\partial \overline{u}}{\partial y} \Rightarrow \nu_t = l_m \cdot v' = l_m^2 \frac{\partial u}{\partial y}$.

Proposing that τ (Shear Stress) ≈ constant and $l_m \propto y$, Prandtl derived the logarithmic law

$$\tau = \mu_t \frac{\partial \overline{u}}{\partial y} = const. \quad \Rightarrow y^2 \left(\frac{\partial \overline{u}}{\partial y} \right)^2 = const.$$

$$\boxed{\overline{u} = c_1 \ln y + c_2}$$

The logarithmic profile is valid away from wall where $\nu_t >> \nu_l$. Very close to wall, $\mu \approx \mu_l$ = const.

$$\mu_l \frac{\partial \bar{u}}{\partial y} = const. \quad \Rightarrow \quad \frac{\partial \bar{u}}{\partial y} = const.$$

$$\boxed{\bar{u} = c\, y}$$

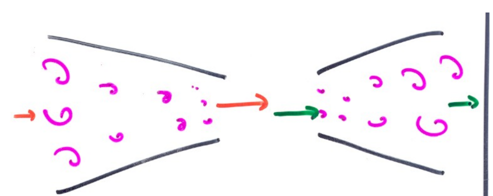
<u>ONE EQUATION MODEL</u>

Define the kinetic energy of the fluctuating motion as $k = \dfrac{u'^2 + v'^2 + w'^2}{2}$. If we take the velocity fluctuation, $v' = \sqrt{k}$, then $\nu_t = l_m \sqrt{k}$. For kinetic energy of turbulence, a transport equation is derived in the form

TURBULENT KINETIC ENERGY EQUATION

$$\underbrace{\frac{\partial k}{\partial t}}_{storage} + \left(u \underbrace{\frac{\partial k}{\partial x} + v \frac{\partial k}{\partial y}}_{convective\ transport\ of\ k} + w \frac{\partial k}{\partial z} \right)$$
$$= \underbrace{\frac{\partial}{\partial x}\left\{ \nu_l + \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial x} \right\} + \frac{\partial}{\partial y}\left\{ \nu_l + \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial y} \right\} + \frac{\partial}{\partial z}\left\{ \nu_l + \frac{\nu_t}{\sigma_k} \frac{\partial k}{\partial z} \right\}}_{diffusive\ transport\ of\ k} + \underbrace{P_k}_{rate\ of\ production\ of\ k} + \underbrace{D_k}_{rate\ of\ dissipation\ of\ k}$$

The production and dissipation terms are suitably formulated. The figure shows the flow in a convergent-divergent nozzle.



Nozzle

(k decreases and main flow kinetic energy increases)

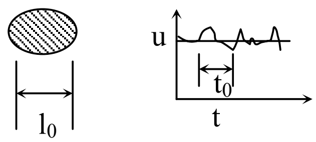
Diffuser

(k increases, main flow decelerates)

k - ε TWO EQUATION MODEL

In addition to the k-equation, a transport equation is derived for the turbulent dissipation rate, ε. Dimensionally, ε ~ k/t_o where t_0 = turbulent time scale, Let l_0 = turbulent length scale and v_0 = turbulent velocity scale

$$\nu_t \sim v_0 \, l_0 \qquad or \quad \frac{l_0^2}{t_0}$$



$$\varepsilon \rightarrow k/t_{0.}, \; v_o \rightarrow \sqrt{k}\;, \; \nu_t \rightarrow \frac{l_0^2}{t_0} = v_0^2 \cdot t_0 \,., \; \nu_t \rightarrow k^2/\varepsilon$$

$$\boxed{\nu_t = c_\mu \cdot \frac{k^2}{\varepsilon}}$$

The transport equation for ε is also similar to that of k. it has a form

TURBULENT DISSIPATION RATE EQUATION

$$\underbrace{\frac{\partial \varepsilon}{\partial t}}_{storage} + \left(u \underbrace{\frac{\partial \varepsilon}{\partial x} + v\frac{\partial \varepsilon}{\partial y}}_{convective \;\; transport} + w\frac{\partial \varepsilon}{\partial z} \right)$$

$$= \underbrace{\frac{\partial}{\partial x}\left\{ \nu_l + \frac{\nu_t}{\sigma_k}\frac{\partial \varepsilon}{\partial x} \right\} + \frac{\partial}{\partial y}\left\{ \nu_l + \frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial y} \right\} + \frac{\partial}{\partial z}\left\{ \nu_l + \frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial z} \right\}}_{diffusive \; transport} + \underbrace{P_\varepsilon}_{rate \; of \;\; production} + \underbrace{D_\varepsilon}_{rate \; of \;\; dissipation}$$

In the k – transport equation, the rate of destruction is $\sim -\frac{\nu_t}{l_0^2} \cdot k = -C_D \frac{k^{3/2}}{l_0} = -\varepsilon$. The production due to viscous dissipation of mean flow KE is given as (in Cartesian Tensor Notation):

$$\underbrace{P_k}_{\text{production rate of turbulent kinetic energy}} = \underbrace{\nu_t \left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right] \frac{\partial \bar{u}_i}{\partial x_j}}_{\text{viscous dissipation of main flow kinetic energy}}$$

For the ε equation

$$P_\varepsilon = C_{\varepsilon 1} \cdot \frac{\varepsilon}{k} \cdot \nu_t \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right) \frac{\partial \bar{u}_i}{\partial x_j}; \quad D_\varepsilon = \frac{\varepsilon}{t_0} = C_{\varepsilon 2} \frac{\varepsilon^2}{k}.$$

In summary, we have

$$\frac{\partial k}{\partial t} + \bar{u}_i \frac{\partial k}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{\nu_l + \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial x_i}\right\} + \nu_t \left[\underbrace{\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}}_{P_k}\right] \frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{\varepsilon}_{D_k}$$

$$\frac{\partial \varepsilon}{\partial t} + \bar{u}_i \frac{\partial \varepsilon}{\partial x_i} = \frac{\partial}{\partial x_i}\left\{\nu_l + \frac{\nu_t}{\sigma_\varepsilon}\frac{\partial \varepsilon}{\partial x_i}\right\} + C_{\varepsilon 1}\frac{\varepsilon}{k} \cdot \nu_t \left[\underbrace{\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}}_{P_\varepsilon}\right] \frac{\partial \bar{u}_i}{\partial x_j} - \underbrace{C_{\varepsilon_2}\frac{\varepsilon^2}{k}}_{D_\varepsilon}$$

The model has five model constants: σ_k, σ_ε, $C_{\varepsilon 1}$, $C_{\varepsilon 2}$, $C\mu$. The k - ε model is an equilibrium, anisotropic model. Equilibrium between convective/diffusive transports and rates of production/destruction is considered. For the above reason it is not valid very close to the wall, initial parts of jets/shear layers/ wakes etc., where non-equilibrium and anisotropic effects may be important. The model is also not valid for problems with strong streamline curvature. Close to wall, (sublayer laminar) the predictions of k-ε model are matched with the universal velocity profile obtained from the mixing length model. This is known as the standard wall function approach. In addition to these, at inlet of the flow domain, the inlet turbulence level has to be specified.