Lecture 11: Constrained Linear Quadratic Optimal Control

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Recommended Reading

- [MLG] Chapter 7 "Constrained Quadratic Optimal Control"
- "Recent Advances in Model Predictive Control and Related Areas" by J. H. Lee and B. Cooley (to be handed out)
- "Linear Systems with State and Control Constraints: The Theory and Application of Maximal Output Admissible Sets" by E. G. Gilbert and K. T. Tan
- "Optimal Infinite-Horizon Feedback Laws for A General Class of Constrained Discrete-Time Systems: Stability and Moving-Horizon Approximations" by S. S. Keerthi and E. G. Gilbert
- "Stability of Constrained Receding Horizon Control," by J. B. Rawlings and K. R. Muske (IEEE TAC 38(10):1512-1516)

Constrained Quadratic Optimal Control

- Consider the same quadratic optimal control problem but with constraints on the input and the state.

- **Synthesis:** Closed-form optimal feedback law cannot be derived (i.e., the dynamic program is not solvable analytically).

  $$\Downarrow$$

  Solve a finite horizon open-loop control problem at each sample time after a feedback update (*a la* MPC).

  MPC can be suboptimal because,

  - *for deterministic systems*, a finite horizon problem is solved as an approximation to an ∞-horizon problem.

  - *for stochastic systems*, an open-loop optimal control problem is solved at each time step to implement a feedback control.

- **Analysis:** The MPC feedback law for the constrained problem is nonlinear $\Rightarrow$ Difficulties in the closed-loop analysis.

- **Challenge:** Formulate the FH problem such that
  - feasibility is maintained.
  - stability is guaranteed.
  - performance is as close to being optimal as possible.

FH Optimal Control Problem for Deterministic System

- Linear Deterministic System:

$$x(k+1) = Ax(k) + Bu(k)$$

- Solve

$$\min_{u(0),\cdots,u(p-1)} \left\{ V_p \triangleq \sum_{i=0}^{p-1} \left[x^T(i)Qx(i) + u^T(i)Ru(i) \right] + x^T(p)Q_t x(p) \right\}$$

with a given $x(0)$ such that

$$\begin{aligned} x(i) &\in X_{fsb}, \quad i = 1, \cdots, p \\ u(i) &\in U_{fsb}, \quad i = 0, \cdots, p-1 \end{aligned}$$

X_{fsb}, U_{fsb} : compact, convex sets that include the origin as an interior point

Example:

$$F\begin{bmatrix} x(i) \\ u(i) \end{bmatrix} \leq g, \qquad g > 0$$

- Representing the state evolution for the entire horizon as $\mathcal{X} = \mathcal{S}^x x(0) + \mathcal{S}^{\mathcal{U}}\mathcal{U}$,

$$\min_{\mathcal{U}} \left[\mathcal{U}^T(\mathcal{S}^{uT}\bar{Q}\mathcal{S}^u + \bar{R})\mathcal{U} + 2\mathcal{U}^T\mathcal{S}^{uT}\bar{Q}\mathcal{S}^x(0)\right]$$

with

$$\begin{aligned} \mathcal{S}^x x_0 + \mathcal{S}^u\mathcal{U} &\in X_{fsb} \times \cdots \times X_{fsb} \\ \mathcal{U} &\in U_{fsb} \times \cdots \times U_{fsb} \end{aligned}$$

- The above is a convex program which is fundamentally tractable.

- If the constraints can be expressed as linear inequalities, the above is a Quadratic Program (QP).

A Feedback Implementation of the Optimal Control Policy

- Solve at each k $(= 0, \cdots, p-1)$,

$$\min_{u(k),\cdots,u(p-1)} \left\{ V_{p-k} \triangleq \sum_{i=k}^{p-1} \left[x^T(i)Qx(i) + u^T(i)Ru(i) \right] + x^T(p)Q_t x(p) \right\}$$

with a given $x(k)$ such that

$$\begin{aligned} x(i) &\in X_{fsb}, \quad i = k+1, \cdots, p \\ u(i) &\in U_{fsb}, \quad i = k, \cdots, p-1 \end{aligned}$$

- The above set of convex programs defines an *implicit* feedback law $\{u(k) = u_k^*(x(k)), k = 0, \cdots, p-1\}$ where $u_k^*(\cdot)$ represents the solution operator to the convex program for the k_{th} time.

- For purely deterministic system, the implicit feedback law results in the same performance as implementing the optimal trajectory in an open-loop manner, but the feedback law is more robust in reality.

IH Optimal Control Problem for Deterministic System

- Solve

$$\min_{u(0),\cdots,u(\infty)} \left\{ \sum_{i=0}^{\infty} \left[x^T(i)Qx(i) + u^T(i)Ru(i) \right] \right\}$$

for given $x(0)$ and constraints

$$\begin{aligned} x(i) &\in X_{fsb}, i = 1, \cdots, \infty \\ u(i) &\in U_{fsb}, i = 0, \cdots, \infty \end{aligned}$$

- The above is an infinite dimensional optimization which cannot be solved.

- Deriving an optimal feedback strategy using dynamic programming is not feasible.

- Suboptimal strategy of MPC: At each k, solve

$$\min_{u(k),\cdots,u(k+p-1)} \quad \left\{ \sum_{i=0}^{p-1} \left[x^T(k+i)Qx(k+i) + u^T(k+i)Ru(k+i) \right] \right.$$
$$\left. + x^T(k+p)Q_t x(k+p) \right\}$$

subject to the model equation constraint and

$$\begin{aligned} x(k+i) &\in X_{fsb}, i = 1, \cdots, p-1 \\ x(k+p) &\in X_{end}, i = 1, \cdots, p-1 \\ u(k+i) &\in U_{fsb}, i = 0, \cdots, p-1 \end{aligned}$$

The above optimization defines an implicit feedback law $u(k) = u^*(x(k))$ where u^* represents the solution operator.

- The choice of p, Q_t and X_{end} turn out to be critical.

Choice I for the Horizon Size and Terminal Weighting Matrix $\Rightarrow$ Optimal ∞-Horizon Regulator (Constrained LQR)

- After a sufficient time, unconstrained optimal quadratic control will become optimal.

- Choose Q_t as the solution to the Riccati equation

$$A^T Q_t A + Q - A^T Q_t B (B^T Q_t B + R)^{-1} B^T Q_t A = Q_t$$

- Choose p sufficiently large such that the FH optimization yields $x(k+p)$ such that the unconstrained LQR is feasible, i.e.,

$$\begin{aligned} u(k+i) = -L_\infty x(k+i) \;\; &\in \;\; U_{fsb} \quad \forall i \geq p \\ x(k+i) \;\; &\in \;\; X_{fsb} \quad \forall i \geq p \end{aligned}$$

for the system $x(k+i+1) = (A - L_\infty B) x(k+i)$.

For the choice of Q_t,

$$\begin{aligned} & x(k+p)^T Q_t x(k+p) \\ = & \min_{u(k+p),\cdots,u(\infty)} \sum_{i=k+p}^{\infty} \left[x^T(k+i) Q x(k+i) + u^T(k+i) R u(k+i) \right] \end{aligned}$$

- Under these choices, the implicit feedback law obtained from the FH problem is in fact the optimal IH regulator for the constrained system.

- Implementation may be difficult as such a choice of p is dependent on $x(k)$ and may be very large.

Choice II $\Rightarrow$ Suboptimal ∞-Horizon Regulator

- Let's look for a way to fix the horizon size p. For this, we assume that unconstrained LQR will be used after a fixed number (p) of time steps.

- Define X_{ma} ("maximal admissible set") as the largest set such that

$$x(0) \in X_{ma} \quad \Rightarrow \quad x(i) \in X_{fsb} \quad \text{and} \quad -L_\infty x(i) \in U_{fsb} \quad \text{for } i \geq 0$$

  for the closed-loop system

$$x(i+1) = (A - BL_\infty)x(i).$$

  Such a set is convex and can be calculated *a priori*.

- Add the constraint

$$x(k+p) \in X_{ma}$$

directly to the optimization.

- The difference between I and II: In I, we let p vary and become large enough so that $x(k+p) \in X_{ma}$ automatically (w/o any artificial constraint). In II, we fix p and impose the constraint that $x(k+p) \in X_{ma}$. II can be suboptimal.

- Properties:
  - the feasibility of the FH problem implies the feasibility of the IH problem where the remainder (beyond $k+p$) is assumed to be controlled by the unconstrained LQR.
  - the optimality may be lost (i.e., if $x(k+p) \in X_{ma}$ is active).
  - feasibility of this constraint still depends on p, even though it may require a much smaller p than before.

Choice III $\Rightarrow$ Suboptimal ∞-Horizon Regulator

- Assume

$$u(k+i) = 0 \text{ for } i \geq p$$

- Under this assumption

$$x(k+p)^T Q_t x(k+p) = \sum_{i=k+p}^{\infty} \left[x^T(k+i) Q x(k+i) + u^T(k+i) R u(k+i) \right]$$

where Q_t is the solution to

$$A^T Q_t A + Q = Q_t.$$

- The input constraint is automatically satisfied beyond $k+p$.

- The state constraint may not be satisfied beyond the horizon. For this, we can

- define a separate constraint horizon $p + h_c$ and impose the state constraint beyond the FH until $k + p + h_c$. h_c is to be chosen large enough that

$$x(j) \in X_{fsb}, j = 0, \cdots, h_c \Rightarrow x(k + h_c + i) \in X_{fsb} \text{ for all } i > 0$$

  for the open-loop system $x(i+1) = Ax(i)$. This way, the satisfaction of the state constraint within the constraint horizon implies the satisfaction in the IH.

- or, define $\hat{X}_{ma}$ representing the maximal set such that

$$x(0) \in \hat{X}_{ma} \Rightarrow x(i) \in X_{fsb} \text{ for } i \geq 0$$

  for the open-loop system $x(i + 1) = Ax(i)$ and impose the constraint that

$$x(k + p) \in \hat{X}_{ma}.$$

  Again, $\hat{X}_{ma}$ is convex and can be computed a priori.

  However, p ensuring the feasibility of the constraint depends on the initial condition. Hence, p must be chosen carefully.

- For an open-loop unstable systems, we will need the constraint

$$T_u x(k+p) = 0$$

where T_u is the operator that extracts the unstable modes. Q_t can be defined with respect to the stable modes only.

Choice IV for the Horizon Size and Terminal Weighting Matrix

- Impose the constraint

$$x(k+p) = 0.$$

- Under this choice, the FH problem is equivalent to the IH problem.

- The system has to be controllable (as opposed to being just stabilizable) for the constraint to be feasible.

- The equality constraint (which takes the place of the inequality constraint earlier, e.g., $x(k+p) \in \hat{X}_{ma}$ in Choice II) is more restrictive and is more difficult to implement.

- The choice of p ensuring the feasibility of the equality constraint is dependent on the initial condition and can be quite large.

Comparison

- In Choice I,
  - make p sufficiently large (for unconstrained LQR to be feasible beyond the horizon)

- In Choice II,
  - impose the constraint $x(k+p) \in X_{ma}$.

- In Choice III,
  - make h_c sufficiently large (for the state constraint beyond the horizon to be feasible in open-loop)
  - or impose the constraint $x(k+p) \in \hat{X}_{ma}$ (along with $T_u x(k+p) = 0$ for unstable systems).

- In Choice IV,
  - impose the constraint $x(k+p) = 0$.

- Choice IV is the most restrictive and the most difficult to implement.

- In comparing Choice I and III, the choice of $p + h_c$ in Case III is generally less restrictive than that of p in Case I. In addition, h_c has less effect on the complexity of the optimization than increasing p.

  On the other hand, the performance of Choice I should be better than that of Choice III in general. For a sufficiently large p, the difference should be minimal, however.

- In Choice II (or in the alternative formulation of Choice III), the maximal admissible set (MAS) needs to be defined. One may need to bound the MAS with a simple-shaped polyhedron and use that instead.

Constraint Softening

- State constraints can be infeasible (regardless of choice of the horizon size).

- We need a method to relax it.

- For example,

$$Fx(k) \leq g$$

$$\Downarrow$$

$$Fx(k) \leq g + \epsilon(k)$$

and solve

$$\min_{u(0),\cdots,u(p-1),\epsilon(0),\cdots,\epsilon(p-1)} \left\{ V_p = \sum_{i=0}^{p-1} \Big[x^T(i)Qx(i) + u^T(i)Ru(i) + \epsilon^T(i)Q_\epsilon(i)\epsilon(i) \Big] + x^T(p)Q_t x(p) \right\}$$

- The slack variable vector $\epsilon(k)$,

  - time-varying vs. time-invariant

  - vector-valued vs. scalar-valued

$$\Downarrow$$

along with the choice of Q_ϵ determines a trade-off between computational complexity, magnitude and duration of constraint violation can be achieved.

- For a finite Q_ϵ, *exact softening* is not possible.

- Constraints that arise from the IH approximation (e.g., $x(k+p) = \hat{X}_{ma}$) cannot be relaxed since they affect the stability. We will see this later.

Output Feedback

- Output Feedback = State Observer (Estimator) + State Feedback MPC

$$\begin{aligned}
\hat{x}(k+1|k+1) &= A\hat{x}(k|k) + Bu(k) \\
&\quad + K\left(y(k) - C(A\hat{x}(k|k) + Bu^*(\hat{x}(k|k)))\right) \\
u(k) &= u^*(\hat{x}(k|k)))
\end{aligned}$$

$u^*(\cdot)$: the state-feedback law defined implicitly through the mathematical program

- It's not automatic that independent stability of state observer and state feedback regulator give closed-loop stability of the output feedback controller. The control law defined is nonlinear and the previous analysis based on the matrix manipulation and eigenvalues no longer holds.

Analysis - State Feedback

- System:

$$x(k+1) = Ax(k) + Bu(k)$$

- Controller:

$$u(k) = u^*(x(k))$$

where $u^*(\cdot)$ is the solution operator for the FH problem (under choice I, II or III).

- The Basic Result (To Be Proven):

If the FH optimization problem at $k = 0$ is feasible and the optimal cost bounded, then the feasibility is maintained and

$$x(k) \to 0 \text{ and } u(k) \to 0 \text{ as } k \to \infty$$

- To exemplify the approach, we will just consider the case of Choice II without soft constraint.

- Under the Choice III, the FH problem we solve at k is

$$\min_{u(k),\cdots,u(k+p-1)} \left\{ \sum_{i=0}^{p-1} \left[x^T(k+i)Qx(k+i) + u^T(k+i)Ru(k+i) \right] + x^T(k+p)Q_t x(k+p) \right\}$$

subject to the model equation constraint and

$$\begin{aligned} x(k+i) &\in X_{fsb}, i = 1, \cdots, p-1 \\ x(k+p) &\in \widehat{X}_{ma} \\ u(k+i) &\in U_{fsb}, i = 0, \cdots, p-1 \end{aligned}$$

and Q_t chosen to satisfy the Lyapunov equation. (In the case of unstable system, we have the extra constraint of $T_u x(k) = 0$).

- This is equivalent to solving the following IH problem:

$$\min_{u(k),\cdots,u(\infty)} \left\{ \sum_{i=0}^{\infty} \left[x^T(k+i)Qx(k+i) + u^T(k+i)Ru(k+i) \right] \right\}$$

subject to the model equation constraint and

$$\begin{aligned} x(k+i) &\in X_{fsb}, \quad i = 1, \cdots, \infty \\ u(k+i) &\in U_{fsb}, \quad i = 0, \cdots, p-1 \\ u(k+i) &= 0, \quad i \geq p \end{aligned}$$

- Denote the optimal cost of the FH problem as $J_p(x(k))$.

- Note that, in order for the FH problem at $k = 0$ to be feasible,

  - the system must be stabilizable.

  - the state constraint should be feasible for a sufficiently large p. (Otherwise use soft constraints.)

- p must be chosen large enough for the FH problem to be feasible.

- **Key Idea:**
  - Denote optimal solution for the FH problem at k as $[u(k|k), u(k+1|k), \cdots, u(k+p-1|k)]$.
  - Note that, $[u(k+1|k), \cdots, u(k+p-1|k), 0]$ is a *feasible* input trajectory for the optimization at $k+1$. Denote the cost from this suboptimal input as $\hat{J}_p(x(k+1))$. Then,

$$J_p(x(k)) = \hat{J}_p(x(k+1)) + x^T(k)Qx(k) + u^T(k)Ru(k)$$

  - Since $J_p(x(k+1)) \leq \hat{J}_p(x(k+1))$,

$$J_p(x(k)) \geq J_p(x(k+1)) + x^T(k)Qx(k) + u^T(k)Ru(k)$$

$$\Downarrow$$

$$J_p(x(0)) - J_p(x(\infty)) \geq \sum_{k=0}^{\infty} x^T(k)Qx(k) + u^T(k)Ru(k)$$

– Since $J_p(x(0)) < \infty$ and $J_p(x(k))$ is lower-bounded by 0, $J_p(x(0)) - J_p(x(\infty))$ is finite. This means, $x(k) \to 0$ and $u(k) \to 0$.

- This proves the *attractivity*. One must prove *Lyapunov stability*, i.e.,

$$\forall \epsilon > 0, \exists \delta(\epsilon) \text{ such that } x(0) in \mathbf{B}_\delta \longrightarrow x(i) in \mathbf{B}_\epsilon,$$

in addition to complete the proof for asymptotic stability (EXTRA CREDIT HW).

- A similar proof can be developed for the case with soft constraints (EXTRA CREDIT HW).

- The proofs for Choice I and III can be based on the same idea.

Analysis - Output Feedback

- Because the MPC regulator $y(k) = u^*(x(k))$ is nonlinear, analysis becomes complicated.

- Independent stability of the state observer and the state feedback does not imply closed-loop stability under the output feedback, in general.

- However, we can prove the stability in the case that $u^*(\cdot)$ is generated by solving QPs.

- The closed-loop system we have can be described as

$$x(k+1) = Ax(k) + Bu^*(\widehat{x}(k))$$

$$\Downarrow$$

$$\hat{x}(k+1) + \hat{x}_e(k+1) = A(\hat{x}(k) + \hat{x}_e(k)) + Bu^*(\hat{x}(k))$$

$$\Downarrow$$

$$\hat{x}(k+1) = \underbrace{A\hat{x}(k) + Bu^*(\hat{x}(k))}_{f(x(k))} + \underbrace{-\hat{x}_e(k+1) + A\hat{x}_e(k)}_{\gamma(k)}$$

- The stability of $\hat{x}(k+1) = f(\hat{x}(k))$ has already been proven.

- Note that $\gamma(k)$ is an exponentially decaying perturbation. Hence, we must show that the stability of the system $\hat{x}(k+1) = f(\hat{x}(k))$ is preserved despite such a perturbation.

- If $f(\cdot)$ is Lipschitz continuous, i.e.,

$$\exists \beta > 0 \text{ such that } \|f(x_1 - x_2)\| \leq \beta \|x_1 - x_2\|$$

  we can show that the stability is preserved. This is true if $u^*(\cdot)$ is Lipschitz continuous.

- For QPs, $u^*(\cdot)$ is Lipschitz continuous.

Stochastic Case - Some Issues

- Solve at each k,

$$\min_{u(k),\cdots,u(k+p-1)} \quad E\left\{\sum_{i=0}^{p-1}\left[x^T(k+i)Qx(k+i)+u^T(k+i)Ru(k+i)\right] + x^T(k+p)Q_t x(k+p)\right\}$$

  with appropriate constraints.

- Input constraints are the same as before.

- State constraints:

  – Constraints on the expected value:

$$E\{x(k+i)\} \in X_{fsb}, \quad i = 1, \cdots, p \tag{1}$$

– Chance constraint:

$$\Pr\{x(k+i) \in X_{fsb}, \quad i = 1, \cdots, p\} \geq \gamma \tag{2}$$

where Pr denotes *probability* and γ is a parameter between 0 and 1.

- With the state constraint on the expected value only, the computation is same as before. In fact, the solution to the above is exactly same as the deterministic case.

- The chance constraint adds to the computational complexity even though the problem is still convex for Gaussian systems.

- Separation theorem no longer holds. Even in the Gaussian case, optimal filter and optimal state regulator do not constitute an optimal output feedback controller.

- The MPC approach is inherently suboptimal in the stochastic case due to the open-loop control assumption.

- Analysis of stability and performance are made much more difficult by the nonlinearity of the MPC control law.